



Differential Geometry II - Smooth Manifolds

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Lecturer: Dr. N. Tsakanikas

Assistant: L. E. Rösler

Exercise Sheet 9 – Solutions

Exercise 1 (to be submitted):

- (a) Let $\pi: E \rightarrow M$ be a smooth vector bundle of rank k over a smooth manifold M . Suppose that $\Phi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ and $\Psi: \pi^{-1}(V) \rightarrow V \times \mathbb{R}^k$ are two smooth local trivializations of E with $U \cap V \neq \emptyset$. Show that the transition function $\tau: U \cap V \rightarrow \text{GL}(k, \mathbb{R})$ between Φ and Ψ is smooth.
- (b) Consider the tangent bundle $\pi: TM \rightarrow M$ of a smooth n -manifold M and let $\Phi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^n$ and $\Psi: \pi^{-1}(V) \rightarrow V \times \mathbb{R}^n$ be the smooth local trivializations of TM associated with two smooth charts (U, φ) and (V, ψ) for M . Determine the transition function $\tau: U \cap V \rightarrow \text{GL}(n, \mathbb{R})$ between Φ and Ψ .
- (c) Consider the tangent bundle $\pi: TS^2 \rightarrow S^2$ of the unit sphere $S^2 \subseteq \mathbb{R}^3$. Compute the transition function associated with the two local trivializations determined by stereographic coordinates.
- (d) Let $\pi: E \rightarrow M$ be a smooth vector bundle of rank k over a smooth manifold M . Suppose that $\{U_\alpha\}_{\alpha \in A}$ is an open cover of M , and that for each $\alpha \in A$ we are given a smooth local trivialization $\Phi_\alpha: \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^k$ of E . For each $\alpha, \beta \in A$ such that $U_\alpha \cap U_\beta \neq \emptyset$, let $\tau_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow \text{GL}(k, \mathbb{R})$ be the transition function between the smooth local trivializations Φ_α and Φ_β . Show that the following identity is satisfied for all $\alpha, \beta, \gamma \in A$:

$$\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p) = \tau_{\alpha\gamma}(p) \quad \text{for all } p \in U_\alpha \cap U_\beta \cap U_\gamma. \quad (\star)$$

Solution:

- (a) Consider the standard bases $\{e_i\}_{i=1}^k$ of \mathbb{R}^k and $\{E_{ij}\}_{i,j=1}^k$ of $M(k, \mathbb{R})$. For each $p \in U \cap V$, denote by $\alpha_{ij}(p) \in \mathbb{R}$ the (i, j) -th element of the matrix $\tau(p) \in \text{GL}(k, \mathbb{R})$ and observe that

$$\tau(p) = \sum_{i,j=1}^k \alpha_{ij}(p) E_{ij}.$$

For each $j \in \{1, \dots, k\}$ we have

$$\tau(p) \cdot e_j = (\alpha_{1j}(p), \dots, \alpha_{kj}(p)) = \sum_{i=1}^k \alpha_{ij}(p) e_i.$$

If now for each $i \in \{1, \dots, k\}$ we denote by π_{ij} the (projection) map

$$\pi_{ij}: (U \cap V) \times \mathbb{R}^k \rightarrow \mathbb{R}, (q, (v_{1j}, \dots, v_{kj})) \mapsto v_{ij},$$

which is smooth by [*Exercise Sheet 3, Exercise 4*], then we obtain

$$(\pi_{ij} \circ \Phi \circ \Psi^{-1})(p, e_j) = \pi_{ij}\left(p, (\alpha_{1j}(p), \dots, \alpha_{kj}(p))\right) = \alpha_{ij}(p).$$

Therefore, each map

$$\alpha_{ij}: U \cap V \rightarrow \mathbb{R}, p \mapsto \alpha_{ij}(p)$$

is smooth as a composite of smooth maps. In view of [*Exercise Sheet 2, Exercise 2(b)*], which gives the smooth chart

$$\psi: \text{GL}(k, \mathbb{R}) \rightarrow \mathbb{R}^{k^2}, \sum_{i,j=1}^k m_{ij} E_{ij} \mapsto \sum_{i,j=1}^k m_{ij} \epsilon_{ij}$$

for $\text{GL}(k, \mathbb{R})$, we now deduce readily that the transition function $\tau: U \cap V \rightarrow \text{GL}(k, \mathbb{R})$ between Φ and Ψ is smooth; indeed, it has a smooth coordinate representation $\psi \circ \tau \circ \varphi^{-1}$ with respect to ψ and any (fixed) smooth chart φ for M around (an arbitrary point) $p \in U \cap V$, since its component functions α_{ij} are smooth.

(b) Denote by (x^1, \dots, x^n) and $(\tilde{x}^1, \dots, \tilde{x}^n)$ the coordinate functions of the smooth coordinate charts (U, φ) and (V, ψ) , respectively, and recall that the associated smooth local trivializations Φ and Ψ , respectively, are defined as follows:

$$\Phi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k, v^i \frac{\partial}{\partial x^i} \Big|_p \mapsto (p, (v^1, \dots, v^n))$$

and

$$\Psi: \pi^{-1}(V) \rightarrow V \times \mathbb{R}^k, \tilde{v}^i \frac{\partial}{\partial \tilde{x}^i} \Big|_p \mapsto (p, (\tilde{v}^1, \dots, \tilde{v}^n)).$$

Since

$$\frac{\partial}{\partial \tilde{x}^i} \Big|_p = \frac{\partial x^j}{\partial \tilde{x}^i}(\hat{p}) \frac{\partial}{\partial x^j} \Big|_p,$$

we have

$$\begin{aligned} (\Phi \circ \Psi^{-1})(p, (v^1, \dots, v^n)) &= \Phi \left(v^i \frac{\partial}{\partial \tilde{x}^i} \Big|_p \right) = \Phi \left(\left(v^i \frac{\partial x^j}{\partial \tilde{x}^i}(\hat{p}) \right) \frac{\partial}{\partial x^j} \Big|_p \right) \\ &= \left(p, \left(v^i \frac{\partial x^1}{\partial \tilde{x}^i}(\hat{p}), \dots, v^i \frac{\partial x^n}{\partial \tilde{x}^i}(\hat{p}) \right) \right) \\ &= \left(p, A_p \cdot (v^1, \dots, v^n)^T \right), \end{aligned}$$

where

$$A_p := \left(\frac{\partial x^j}{\partial \tilde{x}^i}(\hat{p}) \right)_{i,j=1,\dots,n} \in \text{GL}(n, \mathbb{R})$$

is the Jacobian matrix at $\hat{p} = \varphi(p) = \psi(p)$ of the transition map

$$\varphi \circ \psi^{-1}: \psi(U \cap V) \rightarrow \varphi(U \cap V).$$

(Recall also that the matrix A_p represents the differential $d(\varphi \circ \psi^{-1})_{\hat{p}}$ with respect to coordinate bases.) Therefore, the transition function τ between Φ and Ψ is the map

$$\tau: U \cap V \rightarrow \text{GL}(n, \mathbb{R}), \quad p \mapsto A_p = \left(\frac{\partial x^j}{\partial \tilde{x}^i}(\hat{p}) \right)_{i,j}.$$

(c) We use the same notation as the one used in [Exercise Sheet 2, Exercise 5]. According to the solution of part (b), the transition function

$$\tau: \mathbb{S}^2 \setminus \{N, S\} \rightarrow \text{GL}(2, \mathbb{R})$$

between the two smooth local trivializations of $T\mathbb{S}^2$ determined by the stereographic coordinates $(\mathbb{S}^2 \setminus \{N\}, \sigma)$ and $(\mathbb{S}^2 \setminus \{S\}, \tilde{\sigma})$ is given at every point $p \in \mathbb{S}^2 \setminus \{N, S\}$ by the Jacobian matrix at $\hat{p} = \sigma(p) = \tilde{\sigma}(p)$ of the transition map $\sigma \circ \tilde{\sigma}^{-1}$. We saw in [Exercise Sheet 2, Exercise 5] that $\sigma \circ \tilde{\sigma}^{-1}$ is given by the formula

$$(\sigma \circ \tilde{\sigma}^{-1})(\tilde{u}, \tilde{v}) = \left(\frac{\tilde{u}}{\tilde{u}^2 + \tilde{v}^2}, \frac{\tilde{v}}{\tilde{u}^2 + \tilde{v}^2} \right) = (u, v), \quad (\tilde{u}, \tilde{v}) \in \mathbb{R}^2 \setminus \{(0, 0)\}.$$

Thus, its Jacobian at an arbitrary point $(\tilde{u}, \tilde{v}) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ is the matrix

$$J(\sigma \circ \tilde{\sigma}^{-1})(\tilde{u}, \tilde{v}) = \begin{pmatrix} \frac{\tilde{v}^2 - \tilde{u}^2}{(\tilde{u}^2 + \tilde{v}^2)^2} & \frac{-2\tilde{u}\tilde{v}}{(\tilde{u}^2 + \tilde{v}^2)^2} \\ \frac{-2\tilde{u}\tilde{v}}{(\tilde{u}^2 + \tilde{v}^2)^2} & \frac{\tilde{u}^2 - \tilde{v}^2}{(\tilde{u}^2 + \tilde{v}^2)^2} \end{pmatrix}.$$

Note that its determinant equal to $-\frac{1}{(\tilde{u}^2 + \tilde{v}^2)^2}$, which is clearly non-zero.

(d) Fix indices $\alpha, \beta, \gamma \in A$ and a point $p \in U_\alpha \cap U_\beta \cap U_\gamma$. According to Lemma 6.6, for any $v \in \mathbb{R}^k$ we have

$$\begin{aligned} (p, \tau_{\alpha\gamma}(p)v) &= (\Phi_\alpha \circ \Phi_\gamma^{-1})(p, v) \\ &= (\Phi_\alpha \circ \Phi_\beta^{-1}) \circ (\Phi_\beta \circ \Phi_\gamma^{-1})(p, v) \\ &= (\Phi_\alpha \circ \Phi_\beta^{-1})(p, \tau_{\beta\gamma}(p)v) \\ &= (p, \tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p)v), \end{aligned}$$

which implies that

$$\tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p) = \tau_{\alpha\gamma}(p).$$

Since α, β, γ and p were arbitrary, we obtain (\star) .

Exercise 2 (*Smooth vector bundle construction lemma*): Let M be a smooth manifold and let $\{U_\alpha\}_{\alpha \in A}$ be an open cover of M . Suppose that for each $\alpha, \beta \in A$ we are given a smooth map $\tau_{\alpha\beta}: U_\alpha \cap U_\beta \rightarrow \text{GL}(k, \mathbb{R})$ such that (\star) is satisfied for all $\alpha, \beta, \gamma \in A$. Show that there is a smooth vector bundle $E \rightarrow M$ of rank k with smooth local trivializations $\Phi_\alpha: \pi^{-1}(U_\alpha) \rightarrow U_\alpha \times \mathbb{R}^k$ whose transition functions are the given maps $\tau_{\alpha\beta}$.

[Hint: Define an appropriate equivalence relation on $\coprod (U_\alpha \times \mathbb{R}^k)$ and use the vector bundle chart lemma.]

Solution: We first fix some notation: Set

$$\mathcal{E} := \coprod (U_\alpha \times \mathbb{R}^k);$$

and for $(p, v) \in U_\alpha \times \mathbb{R}^k$, denote by $(p, v)_\alpha$ its image in \mathcal{E} .

As suggested by the hint, consider now the following relation \sim on \mathcal{E} : two points $(p, v)_\alpha, (p', v')_\beta \in \mathcal{E}$ are equivalent if and only if

$$p = p' \quad \text{and} \quad v = \tau_{\alpha\beta}(p) \cdot v',$$

in which case we write $(p, v)_\alpha \sim (p', v')_\beta$. Let us check that \sim indeed an equivalence relation on \mathcal{E} :

- *Reflexivity:* By applying (\star) to $\alpha = \beta = \gamma$ we obtain $\tau_{\alpha\alpha} \equiv \text{Id}_{k \times k}$. It follows that $v = \tau_{\alpha\alpha}(p)v$ for all $(p, v) \in U_\alpha \times \mathbb{R}^k$, and thus $(p, v)_\alpha \sim (p, v)_\alpha$.
- *Symmetry:* Suppose that $(p, v)_\alpha \sim (p', v')_\beta$, i.e., $p = p'$ and $v = \tau_{\alpha\beta}(p)v'$. By applying (\star) to α, β and $\gamma = \alpha$ we obtain $\tau_{\beta\alpha}(p) = (\tau_{\alpha\beta}(p))^{-1}$. Thus,

$$v' = (\tau_{\alpha\beta}(p))^{-1} \cdot v = \tau_{\beta\alpha}(p) \cdot v,$$

whence $(p', v')_\beta \sim (p, v)_\alpha$.

- *Transitivity:* Suppose that $(p, v)_\alpha \sim (p', v')_\beta$ and $(p', v')_\beta \sim (p'', v'')_\gamma$. Then

$$p = p' = p'' \quad \text{and} \quad v = \tau_{\alpha\beta}(p) \cdot v', \quad v' = \tau_{\beta\gamma}(p') \cdot v''.$$

In particular, we obtain

$$v = \tau_{\alpha\beta}(p)\tau_{\beta\gamma}(p') \cdot v'' \stackrel{(\star)}{=} \tau_{\alpha\gamma}(p) \cdot v'',$$

which shows that $(p, v)_\alpha \sim (p'', v'')_\gamma$.

Next, set

$$E := \mathcal{E} / \sim$$

and denote by $[(p, v)_\alpha] \in E$ the equivalence class of $(p, v)_\alpha \in \mathcal{E}$. Note that the map $\mathcal{E} \rightarrow M$ sending $(p, v)_\alpha$ to p factors through E , because if $(p, v)_\alpha \sim (p', v')_\beta$, then in particular $p = p'$. So, consider the map

$$\pi: E \rightarrow M, \quad [(p, v)_\alpha] \mapsto p.$$

Now, let α be arbitrary and let us verify that

$$\begin{aligned}\Psi_\alpha: U_\alpha \times \mathbb{R}^k &\rightarrow \pi^{-1}(U_\alpha) \\ (p, v) &\mapsto [(p, v)_\alpha]\end{aligned}$$

is a bijection. For injectivity, suppose that $\Psi_\alpha(p, v) = \Psi_\alpha(p', v')$. In particular, we obtain $p = p'$ and $v = \tau_{\alpha\alpha}(p)v' = v'$. For surjectivity, let $[(p', v')_\beta] \in \pi^{-1}(U_\alpha)$ be arbitrary. Notice that $p := \pi([(p', v')_\beta]) \in U_\alpha$, and set $v = \tau_{\alpha\beta}(p)v'$. Then we have $(p, v)_\alpha \sim (p', v')_\beta$, and thus $[(p', v')_\beta] = \Psi_\alpha(p, v)$. Hence, Ψ_α is bijective, as claimed. Finally, with a similar argument it is straightforward to check that $\Psi_\alpha(\{p\} \times \mathbb{R}^k) = \pi^{-1}(p)$.

By bijectivity we may write $\Phi_\alpha = (\Psi_\alpha)^{-1}$. To endow the fibers $\pi^{-1}(p)$ with a vector space structure, let α_p be such that $p \in U_{\alpha_p}$. We endow $\pi^{-1}(p)$ with the structure of a k -dimensional real vector space via the bijection $\pi^{-1}(p) \cong \{p\} \times \mathbb{R}^k$ provided by Φ_{α_p} . We denote the resulting real vector space by $E_p = \pi^{-1}(p)$. Since we chose α_p at random, we have to check that the choice does not matter. To this end, let α be arbitrary and take $p \in U_\alpha$. We have to check that $\Phi_\alpha|_{E_p}$ is a vector space isomorphism from E_p to $\{p\} \times \mathbb{R}^k$. So, pick $(p, v)_{\alpha_p} \in E_p$, and set $v' = \tau_{\alpha, \alpha_p}(p) \cdot v$, so that $(p, v')_\alpha \sim (p, v)_{\alpha_p}$. Then

$$\Phi_\alpha([(p, v)_{\alpha_p}]) = \Phi_\alpha([(p, v')_\alpha]) = (p, v') = (p, \tau_{\alpha, \alpha_p}(p) \cdot v).$$

As $\tau_{\alpha, \alpha_p}(p) \in \text{GL}(k, \mathbb{R})$, we infer that $\Phi_\alpha|_{E_p} : E_p \rightarrow \{p\} \times \mathbb{R}^k$ is an isomorphism of real vector spaces.

Finally, to apply the Vector Bundle Chart Lemma, we have to verify that the Φ_α 's are compatible. Let α, β be such that $U_\alpha \cap U_\beta \neq \emptyset$. Take $(p, v) \in (U_\alpha \cap U_\beta) \times \mathbb{R}^k$. We want to compute $(\Phi_\alpha \circ \Phi_\beta^{-1})(p, v)$. By construction, we see that $\Phi_\beta^{-1}(p, v) = [(p, v)_\beta]$. Now, let $v' = \tau_{\alpha\beta}(p) \cdot v$, so that $(p, v')_\alpha \sim (p, v)_\beta$. Then we have

$$(\Phi_\alpha \circ \Phi_\beta^{-1})(p, v) = \Phi_\alpha([(p, v')_\alpha]) = (p, v') = (p, \tau_{\alpha\beta}(p) \cdot v).$$

Since by hypothesis the maps $\tau_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow \text{GL}(k, \mathbb{R})$ are smooth, the Vector Bundle Chart Lemma implies that E has a unique topology and smooth structure such that $\pi : E \rightarrow M$ is a smooth vector bundle of rank k , and the Φ_α 's are its local trivializations, with transition functions the $\tau_{\alpha\beta}$'s.

Exercise 3:

(a) Show that the zero section of every smooth vector bundle is smooth.

[Hint: Consider $\Phi \circ \zeta$, where Φ is a smooth local trivialization.]

(b) Let $\pi : E \rightarrow M$ be a smooth vector bundle. Show that if $f, g \in C^\infty(M)$ and if $\sigma, \tau \in \Gamma(E)$, then $f\sigma + g\tau \in \Gamma(E)$.

[Hint: Consider $\Phi \circ (f\sigma + g\tau)$, where Φ is a smooth local trivialization of E .]

(c) Let $E := M \times \mathbb{R}^k$ be a product bundle over a topological manifold M . Show that there is a natural one-to-one correspondence between (continuous) sections of E and continuous functions from M to \mathbb{R}^k .

Moreover, if M is a smooth manifold, show that this is a one-to-one correspondence between smooth sections of E and smooth functions from M to \mathbb{R}^k . Deduce that there is a natural identification between the space $C^\infty(M)$ and the space of smooth sections of the trivial line bundle $M \times \mathbb{R} \rightarrow M$.

Solution:

(a) Let $p \in M$ and let $\Phi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ be a smooth local trivialization, where U is a neighborhood of p . Given $q \in U$, write 0_q for the zero element of $E_q = \pi^{-1}(q)$. By definition we have $\zeta(q) = 0_q \in E_q$. Since $\Phi|_{E_q}: E_q \rightarrow \{q\} \times \mathbb{R}^k$ is a vector space isomorphism, we obtain

$$\Phi(\zeta(q)) = \Phi|_{E_q}(0_q) = 0_{\{q\} \times \mathbb{R}^k} = (q, 0).$$

Hence, $\Phi \circ \zeta|_U = \text{Id}_U \times \mathbb{0}_{\mathbb{R}^k}$ is smooth by [Exercise Sheet 3, Exercise 4]. As Φ is a diffeomorphism, we infer that $\zeta|_U$ is smooth, and as p was arbitrary, we conclude that ζ is smooth.

Remark. Arguing as above (essentially replacing the words “smooth” with “continuous” and “diffeomorphism” with “homeomorphism”), we can also show that, more generally, the zero section of a (topological) vector bundle is continuous.

(b) Let $p \in M$ and let $\Phi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ be a smooth local trivialization, where U is an open neighborhood of $p \in M$. For $q \in U$, denote by $+_q$ the addition and by \cdot_q the scalar multiplication of E_q . By definition we have

$$(f\sigma + g\tau)(q) = f(q) \cdot_q \sigma(q) +_q g(q) \cdot_q \tau(q) \in E_q.$$

Since $\Phi|_{E_q}: E_q \rightarrow \{q\} \times \mathbb{R}^k$ is a vector space isomorphism, we obtain

$$\Phi \circ (f\sigma + g\tau)(q) = f(q)(\Phi|_{E_q})(\sigma(q)) + g(q)(\Phi|_{E_q})(\tau(q)) \in \{q\} \times \mathbb{R}^k.$$

According to [Exercise Sheet 3, Exercise 4], showing that the map $\Phi \circ (f\sigma + g\tau)$ is smooth is equivalent to checking that its post-composition with both projections $\text{pr}_1: U \times \mathbb{R}^k \rightarrow U$ and $\text{pr}_2: U \times \mathbb{R}^k \rightarrow \mathbb{R}^k$ is smooth. By the above formula we obtain

$$\text{pr}_1 \circ \Phi \circ (f\sigma + g\tau) = \text{Id}_U,$$

so it remains to check post-composition with pr_2 . To this end, set $\hat{\sigma} := \text{pr}_2 \circ \Phi \circ \sigma$ and $\hat{\tau} := \text{pr}_2 \circ \Phi \circ \tau$, and note that both of them are smooth functions from U to \mathbb{R}^k . The above formula gives

$$\text{pr}_2 \circ \Phi \circ (f\sigma + g\tau)(q) = f(q)\hat{\sigma}(q) + g(q)\hat{\tau}(q).$$

Due to the smoothness of the maps involved, this is also smooth. Thus, $\Phi \circ (f\sigma + g\tau)|_U$ is smooth, and as Φ is a diffeomorphism, we infer that $(f\sigma + g\tau)|_U$ is smooth. Since $p \in M$ was arbitrary, we conclude that $f\sigma + g\tau$ is a smooth global section of E .

(c) Consider the two projection maps of the given product bundle:

$$\pi = \pi_M: E = M \times \mathbb{R}^k \rightarrow M, (p, v) \mapsto p,$$

and

$$\pi_{\mathbb{R}^k}: E = M \times \mathbb{R}^k \rightarrow \mathbb{R}^k, (p, v) \mapsto v.$$

Note that they are both continuous.

Now, let $f: M \rightarrow \mathbb{R}^k$ be a continuous function. Consider the continuous map

$$\sigma_f: M \rightarrow E, \quad \sigma_f(p) = (p, f(p))$$

and observe that

$$(\pi \circ \sigma_f)(p) = p = \text{Id}_M(p),$$

so σ_f is a global section of E . Conversely, if $\sigma: M \rightarrow E = M \times \mathbb{R}^k$ is a global section of E , then $f_\sigma := \pi_{\mathbb{R}^k} \circ \sigma: M \rightarrow \mathbb{R}^k$ is a continuous map. Finally, it is easy to check that the assignments $f \mapsto \sigma_f$ and $\sigma \mapsto f_\sigma$ just described are inverse to each other; in other words, we have $\sigma = \sigma_{f_\sigma}$ and $f = f_{\sigma_f}$.

If M is a smooth manifold, and hence $\pi: E = M \times \mathbb{R}^k \rightarrow M$ is a smooth product bundle of rank k over M , then the above construction yields an one-to-one correspondence between smooth sections of E and smooth functions from M to \mathbb{R}^k , taking into account *Exercise 3* and *Exercise 4* from *Exercise Sheet 3*. In particular, if $k = 1$, then there is a natural identification between the space $C^\infty(M)$ of smooth functions on M and the space of smooth sections of the trivial smooth line bundle $M \times \mathbb{R} \rightarrow M$.

Remark. Let $\pi: E \rightarrow M$ be a smooth vector bundle. According to *Exercise 3*, the set $\Gamma(E)$ of smooth global sections of E is an \mathbb{R} -vector space and a module over the ring $C^\infty(M)$. Moreover, it follows from *Exercises 2.21* and *6.12* that $\dim_{\mathbb{R}} \Gamma(E) = +\infty$.

Exercise 4 (*Completion of smooth local frames for smooth vector bundles*): Let $\pi: E \rightarrow M$ be a smooth vector bundle of rank k over a smooth manifold M . Prove the following assertions:

- (a) If $(\sigma_1, \dots, \sigma_m)$ is a linearly independent m -tuple of smooth local sections of E over an open subset $U \subseteq M$, where $1 \leq m < k$, then for each $p \in U$ there exist smooth sections $\sigma_{m+1}, \dots, \sigma_k$ of E defined on some neighborhood V of p such that $(\sigma_1, \dots, \sigma_k)$ is a smooth local frame for E over $U \cap V$.
- (b) If (v_1, \dots, v_m) is a linearly independent m -tuple of elements of the fiber E_p for some $p \in M$, where $1 \leq m < k$, then there exists a smooth local frame $(\sigma_1, \dots, \sigma_k)$ for E over some neighborhood of p such that $\sigma_i(p) = v_i$ for every $1 \leq i \leq m$.
- (c) If $A \subseteq M$ is a closed subset and if (τ_1, \dots, τ_k) is a linearly independent k -tuple of sections of $E|_A$ which are smooth in the sense described in *Lemma 6.11*, then there exists a smooth local frame $(\sigma_1, \dots, \sigma_k)$ for E over some neighborhood of A such that $\sigma_i|_A = \tau_i$ for every $1 \leq i \leq k$.

[Hint: Use *Lemma 6.11*.]

Solution:

- (a) Let V_0 be an open neighborhood of p in M over which there exists a smooth local trivialization $\Phi: \pi^{-1}(V_0) \rightarrow V_0 \times \mathbb{R}^k$ of E . As $\Phi(\sigma_1(p)), \dots, \Phi(\sigma_m(p)) \in \{p\} \times \mathbb{R}^k$ are linearly independent, there are vectors $v_{m+1}, \dots, v_k \in \mathbb{R}^k$ such that the set

$$\{\Phi(\sigma_1(p)), \dots, \Phi(\sigma_m(p)), (p, v_{m+1}), \dots, (p, v_k)\}$$

is a basis of $\{p\} \times \mathbb{R}^k \cong \mathbb{R}^k$. For each $m + 1 \leq i \leq k$, define

$$\sigma_i: V_0 \rightarrow E, \sigma_i(q) := \Phi^{-1}(q, v_i)$$

and note that σ_i is smooth, as both $q \mapsto (q, v_i)$ and Φ^{-1} are so. Now, consider the function

$$d: V_0 \rightarrow \mathbb{R}, q \mapsto \det \left(\text{pr}_2 \left(\Phi(\sigma_1(q)) \right), \dots, \text{pr}_2 \left(\Phi(\sigma_k(q)) \right) \right).$$

We have $d(p) \neq 0$, since by construction the set

$$\left\{ \text{pr}_2 \left(\Phi(\sigma_1(p)) \right), \dots, \text{pr}_2 \left(\Phi(\sigma_k(p)) \right) \right\}$$

is a basis of \mathbb{R}^k . As d is continuous, there exists a neighborhood V of p such that $d|_V$ is nowhere zero. Hence, $(\sigma_1, \dots, \sigma_k)$ is a smooth local frame for E over $U \cap V$.

(b) We may complete (v_1, \dots, v_m) to a basis (v_1, \dots, v_k) of $E_p \cong \mathbb{R}^k$. Let U be an open neighborhood of $p \in M$ such that there exists a smooth local trivialization $\Phi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ of E over U . As in part (a), we define $\sigma_i: U \rightarrow E$ by $\sigma_i(q) := \Phi^{-1}(q, v_i)$, and again by continuity of the determinant, this gives a smooth local frame on some open neighborhood $V \subseteq U$ of p .

(c) By hypothesis and by *Lemma 6.11* (applied for $U = M$), for each $i \in \{1, \dots, k\}$ there exists a smooth global section τ_i of E such that $\tau_i|_A = \sigma_i$. Therefore, for every $p \in A$ the set $\{\tau_1(p), \dots, \tau_k(p)\}$ is a basis of E_p , and by continuity of the determinant there exists an open neighborhood U_p of p in M such that $\{\tau_1(q), \dots, \tau_k(q)\}$ is a basis of E_q for each $q \in U_p$. Thus, $U := \bigcup_{p \in A} U_p$ is an open subset of M containing A and additionally for every $x \in U$ the set $\{\tau_1(x), \dots, \tau_k(x)\}$ is a basis of the fiber E_x ; in other words, (τ_1, \dots, τ_k) is a smooth local frame for E over the open neighborhood U of A .

Exercise 5 (*Correspondence between smooth local frames and smooth local trivializations*): Let $\pi: E \rightarrow M$ be a smooth vector bundle of rank k over a smooth n -manifold M .

(a) Given a smooth local trivialization $\Phi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ of E over U , construct a smooth local frame (σ_i) for E over U . (We say that the smooth local frame (σ_i) is associated with the smooth local trivialization Φ .)

(b) Show that every smooth local frame (σ_i) for E is associated with a smooth local trivialization Φ of E .

[Hint: Define the inverse of Φ using (σ_i) and show that it is a bijective local diffeomorphism to conclude.]

(c) Deduce that E is smoothly trivial if and only if it admits a smooth global frame. Interpret this result in case that E is a smooth *line bundle*, i.e., when $k = 1$.

(d) Let (U, φ) be a smooth coordinate chart for M with coordinate functions (x^i) and assume that there exists a smooth local frame (σ_i) for E over U . Consider the map

$$\tilde{\varphi}: \pi^{-1}(U) \rightarrow \varphi(U) \times \mathbb{R}^k, v^i \sigma_i(p) \mapsto (x^1(p), \dots, x^n(p), v^1, \dots, v^k).$$

Show that $(\pi^{-1}(U), \tilde{\varphi})$ is a smooth coordinate chart for E .

Solution:

(a) Let e_1, \dots, e_k be the standard basis of \mathbb{R}^k . As in *Exercise 4(b)*, the smooth local sections $\sigma_i: U \rightarrow E$ defined by $\sigma_i(q) = \Phi^{-1}(q, e_i)$ determine a smooth local frame for E over U .

(b) Let (σ_i) be a smooth local frame for E over an open subset U of M . Consider the map

$$\Psi: U \times \mathbb{R}^k \rightarrow \pi^{-1}(U), \quad \Psi(q, v) := v_1 \cdot_q \sigma_1(q) + \dots + v_k \cdot_q \sigma_k(q) \in E_q,$$

where $v = \sum v^i e_i \in \mathbb{R}^k$ is arbitrary. It is straightforward to check that $\pi \circ \Psi = \text{pr}_1$.

Let us first show that Ψ is bijective. To prove its injectivity, suppose that $\Psi(q, v) = \Psi(q', v')$. By applying π we see that $q = q'$, and thus

$$v_1 \cdot_q \sigma_1(q) + \dots + v_k \cdot_q \sigma_k(q) = v'_1 \cdot_q \sigma_1(q) + \dots + v'_k \cdot_q \sigma_k(q)$$

inside E_q . As $\sigma_1(q), \dots, \sigma_k(q)$ is a basis of E_q , we infer that $v = v'$, and thus we establish the injectivity of Ψ . Now, to prove the surjectivity of Ψ , let $e \in \pi^{-1}(U)$ be arbitrary. Set $q = \pi(e)$ and let $v = (v_1, \dots, v_k)$ be such that

$$e = v_1 \cdot_q \sigma_1(q) + \dots + v_k \cdot_q \sigma_k(q)$$

inside E_q . Then $e = \Psi(q, v)$, so we are done.

It remains to check that Ψ is a local diffeomorphism. Let $p \in U$ and let $\Phi: \pi^{-1}(V) \rightarrow V \times \mathbb{R}^k$ be a smooth local trivialization of E , where V is an open neighborhood of p contained in U . Since Φ is a diffeomorphism, if we could show that $\Phi \circ \Psi|_{V \times \mathbb{R}^k}$ is a diffeomorphism from $V \times \mathbb{R}^k$ to itself, then we would infer that $\Psi|_{V \times \mathbb{R}^k}$ is a diffeomorphism from $V \times \mathbb{R}^k$ to its image $\pi^{-1}(V)$.

Since $\Phi \circ \sigma_i|_V: V \rightarrow V \times \mathbb{R}^k$ is smooth and since post-composition with pr_1 equals Id_V , we see that it is of the form

$$\Phi \circ \sigma_i|_V(q) = \left(q, (\sigma_i^1(q), \dots, \sigma_i^k(q)) \right)$$

for some smooth functions $\sigma_i^1, \dots, \sigma_i^k: V \rightarrow \mathbb{R}$. If we denote by $A: V \rightarrow \text{Mat}(k \times k, \mathbb{R})$ the function sending q to the matrix $(\sigma_i^j(q))_{1 \leq j, i \leq k}$ (where j is the index for the lines and i is the index for the columns of the matrix), then A is smooth, as every component is smooth. Furthermore, the image of A lies in $\text{GL}(k, \mathbb{R})$ because $\sigma_1(q), \dots, \sigma_k(q)$ is a basis of E_q by assumption. Now, by construction of Ψ , it is straightforward to check that for any $(q, v) \in V \times \mathbb{R}^k$ we have

$$(\Phi \circ \Psi)(q, v) = (q, A(q) \cdot v) \in V \times \mathbb{R}^k.$$

This is clearly smooth, as A is smooth. We then also see that $(\Phi \circ \Psi|_{V \times \mathbb{R}^k})^{-1}$ sends (q, v) to $(q, A(q)^{-1} \cdot v)$, which is smooth as well – we use here that the map $\text{GL}(k, \mathbb{R}) \rightarrow \text{GL}(k, \mathbb{R})$ sending a matrix to its inverse is smooth. Therefore, $\Psi|_{V \times \mathbb{R}^k}: V \times \mathbb{R}^k \rightarrow \pi^{-1}(V)$ is a diffeomorphism, as desired.

In conclusion, Ψ is a bijective local diffeomorphism, and hence a global diffeomorphism by *Proposition 4.9(f)*. It is now straightforward to check that $\Phi = \Psi^{-1}: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ is a smooth local trivialization of E over U such that the given smooth local frame (σ_i) is associated with Φ .

(c) Recall that E is smoothly trivial if and only if it admits a smooth global trivialization. Thus, E is smoothly trivial if and only if it admits a smooth global frame by (a) and (b).

Assume now that E is a smooth vector bundle of rank $k = 1$ over M . Then E is smoothly trivial if and only if it admits a smooth global frame. Such a frame consists of a single smooth global section $\sigma: M \rightarrow E$ with the property that for each $p \in M$, the element $\sigma(p) \in E_p$ is a basis of the 1-dimensional \mathbb{R} -vector space E_p , and hence $\sigma(p) \neq 0$. Conversely, every smooth global section $\sigma: M \rightarrow E$ of E such that $\sigma(p) \in E_p \setminus \{0\}$ determines a smooth global frame for E . In conclusion, the smooth *line bundle* $E \rightarrow M$ is smoothly trivial if and only if it admits a *nowhere vanishing* smooth global section.

(d) By part (b) there exists a smooth local trivialization $\Phi: \pi^{-1}(U) \rightarrow U \times \mathbb{R}^k$ such that (σ_i) is associated with Φ . In particular, we have $(\Phi \circ \sigma_i)(q) = (q, e_i)$ for all $q \in U$, where e_1, \dots, e_k is the standard basis of \mathbb{R}^k . Now let $\psi: \pi^{-1}(U) \rightarrow \varphi(U) \times \mathbb{R}^k$ be the composition $\psi = (\varphi \times \text{Id}_{\mathbb{R}^k}) \circ \Phi$, where $\varphi \times \text{Id}_{\mathbb{R}^k}: U \times \mathbb{R}^k \rightarrow \varphi(U) \times \mathbb{R}^k$ is defined by applying φ on U and $\text{Id}_{\mathbb{R}^k}$ on \mathbb{R}^k . Note that ψ is a diffeomorphism as both Φ and $\varphi \times \text{Id}_{\mathbb{R}^k}$ are diffeomorphisms. To see how ψ acts on the points of $\pi^{-1}(U)$, let $e \in \pi^{-1}(U)$ be arbitrary, and set $q = \pi(e) \in U$. As $\sigma_1(q), \dots, \sigma_k(q)$ is a basis of E_q , there exist real numbers v^1, \dots, v^k such that $e = \sum v^i \sigma_i(q)$. As $(\Phi \circ \sigma_i)(q) = (q, e_i)$, we obtain $\Phi(e) = (q, (v^i))$, so $\psi(e) = (\varphi(q), (v^i))$. Therefore, $\psi = \tilde{\varphi}$. As ψ is a diffeomorphism, this proves that $(\pi^{-1}(U), \tilde{\varphi})$ is a smooth coordinate chart for E .

Remark. By arguing as in the solution to *Exercise 5(a)(b)* – essentially by replacing the words “smooth” with “continuous” and “diffeomorphism” with “homeomorphism” – one can also show that, more generally, there is a correspondence between (continuous) local frames and (continuous) local trivializations for any (topological) vector bundle. This allows one to prove the topological case of *Proposition 6.15* with an essentially identical argument to the smooth case (which was treated in the lecture).

Exercise 6 (*Uniqueness of the smooth structure on TM*): Let M be a smooth n -manifold. Show that the topology and smooth structure on the tangent bundle TM constructed in *Proposition 3.12* are the unique ones with respect to which $\pi: TM \rightarrow M$ is a smooth vector bundle with the given vector space structure on the fibers, and such that all coordinate vector fields are smooth local sections.

[Hint: Use *Exercise 5(d)*.]

Solution: Denote by $(TM)'$ a smooth manifold with underlying set $TM = \bigsqcup_{p \in M} T_p M$, but with possibly different topology and smooth structure than the ones on the usual tangent bundle TM , such that the map

$$\pi': (TM)' \rightarrow M, \quad v \in T_p M \mapsto p \in M$$

gives $(TM)'$ the structure of a smooth vector bundle over M such that all the coordinate vector fields are smooth local sections. (Note that, as set-theoretic maps, π and π' are the same, but we denote them differently to emphasize that their source may have different topology and smooth structure.) By assumption, if $(U, (x^1, \dots, x^n))$ is a smooth chart for M , then the maps

$$\frac{\partial}{\partial x^i}: U \rightarrow (TM)', \quad p \mapsto \frac{\partial}{\partial x^i} \Big|_p$$

are smooth local sections of π' . Since for every $p \in U$ the vectors

$$\left. \frac{\partial}{\partial x^1} \right|_p, \dots, \left. \frac{\partial}{\partial x^n} \right|_p$$

form a basis of $T_p M$, it follows that $\left(\left. \frac{\partial}{\partial x^i} \right|_p \right)_{1 \leq i \leq n}$ is a smooth local frame for $(TM)'$, and according to *Exercise 5(d)*, the map

$$\begin{aligned} \tilde{\varphi}: (\pi')^{-1}(U) &\rightarrow \varphi(U) \times \mathbb{R}^n \\ \left(p, v^i \left. \frac{\partial}{\partial x^i} \right|_p \right) &\mapsto (\varphi(p), v^1, \dots, v^n) \end{aligned}$$

is a smooth chart for $(TM)'$. But the same holds for TM (as we saw in the proof of *Proposition 3.12*). It follows that the identity map $TM \rightarrow (TM)'$ is a diffeomorphism. In particular, it is a homeomorphism, and thus also the topology agrees.

Remark. We somewhat used in *Exercise 6* that the smooth structure actually determines the topology. That is, we have the following:

Let M be a set and let \mathcal{T} and \mathcal{T}' be two topologies on M , both endowing it with the structure of a topological manifold. Supposed that \mathcal{A} is an atlas for both topologies, such that both $(M, \mathcal{T}, \mathcal{A})$ and $(M, \mathcal{T}', \mathcal{A})$ are smooth manifolds. Then $\mathcal{T} = \mathcal{T}'$. Indeed, the identity $\text{Id}_M: (M, \mathcal{T}, \mathcal{A}) \rightarrow (M, \mathcal{T}', \mathcal{A})$ is smooth, as we have the same atlas on both sides; in particular, it is continuous, so $\mathcal{T}' \subseteq \mathcal{T}$. A symmetric argument also shows that the reverse inclusion holds. Therefore, $\mathcal{T} = \mathcal{T}'$, as claimed.